SIMPLE MODULES WITH CHARACTER HEIGHT AT MOST ONE FOR THE RESTRICTED WITT ALGEBRAS

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Abstract. The simple modules with character height at most one for the restricted Witt algebras are considered. Their classification, construction, and dimension formulas are reduced to the same for the general linear algebra. Results of Chang and Shen are recovered in the process.

0. Introduction

Fix $n \geq 1$ and let $W = W(n, \mathbf{1})$ be the restricted Witt algebra over an algebraically closed field F of characteristic $p \geq 5$. Let $\chi \in W^* = \operatorname{Hom}_F(W, F)$. A W-module M *has character* χ provided

$$D^p \cdot m - D^{[p]} \cdot m = \chi(D)^p m$$

for all $D \in W$, $m \in M$. Not every module has a character, but at least every simple module has one [SF, Theorem 2.5, p. 207]. Note that restricted modules have the character $\chi = 0$.

Let $W = \dot{\sum}_i W_i$ be the standard grading on W and put $W^i = \sum_{j \geq i} W_j$. Implicit in Chang's work [C], but first defined for the algebra $W(1, \mathbf{1})$ by Strade in [St], is the useful notion of the *height* $\operatorname{ht}\chi$ of the character χ:

$$\operatorname{ht}\chi = \min\{i \geq -1 \mid \chi(W^i) = 0\}.$$

In this paper, we classify the simple W-modules having character χ of height at most one (i.e., with $\operatorname{ht}\chi \in \{-1, 0, 1\}$) and compute their dimensions (up to the corresponding data for the simple modules for $W_0 \cong \mathfrak{gl}_n(F)$). The determination of the simple restricted modules (i.e., the case $\operatorname{ht}\chi = -1$) was completed by Shen in [Sh]. Also, Chang [C] constructed the simple modules (with arbitrary character) for $W(1, \mathbf{1})$ and Koreshkov [K] obtained some results in regard to the simple modules for $W(2, \mathbf{1})$. We give a uniform treatment of the three cases $\operatorname{ht}\chi = -1, 0, 1$ and recover some of these authors' results in the process.

1991 *Mathematics Subject Classification.* Primary 17B50; Secondary 17B10.
Key words and phrases. Witt algebra, Cartan-type Lie algebras, simple nonrestricted modules.

We begin in Section 1 by determining a convenient representative for the orbit of χ under the conjugacy action of the group of (restricted) automorphisms of W. For our purposes, χ may often be replaced by this character, with the advantage being that certain arguments are thus simplified. In Section 2, we show that most of the simple modules are obtained by starting with a simple W_0-module, extending the action trivially to W^1, and inducing to W. (It is this construction that requires our main assumption $\operatorname{ht}\chi \leq 1$.) In Section 3 we consider the few exceptional cases where these induced modules are not simple and determine their structures by identifying them with the terms in the usual de Rham complex for W (or a modified such complex in the case $\operatorname{ht}\chi = 0$). Finally, in Section 4 we assemble the results and state the main theorems.

I thank Dan Nakano for a couple of conversations that prompted this research. I also thank the referee for some helpful suggestions.

1. Character Orbit Representatives

Let $a, b \in \mathbb{Z}^n$. We write $a \leq b$ if $a_i \leq b_i$ for all $1 \leq i \leq n$ and we write $a < b$ if $a \leq b$ but $a \neq b$. If $a, b \geq 0$, define $\binom{a}{b} = \prod_i \binom{a_i}{b_i}$, where $\binom{a_i}{b_i}$ is the usual binomial coefficient with the convention that $\binom{a_i}{b_i} = 0$ unless $b_i \leq a_i$. Set $A = \{a \in \mathbb{Z}^n \mid 0 \leq a \leq \tau\}$, where $\tau := (p-1, \dots, p-1)$. The *divided power algebra* $\mathfrak{A}$ is the associative F-algebra having F-basis $\{x^{(a)} \mid a \in A\}$ and multiplication subject to the rule

$$x^{(a)} x^{(b)} = \binom{a+b}{a} x^{(a+b)},$$

where $x^{(c)} := 0$ if $c \notin A$.

For each $1 \leq i \leq n$, let D_i denote the derivation of $\mathfrak{A}$ uniquely determined by the property $D_i x^{(a)} = x^{(a-\epsilon_i)}$, where ϵ_i is the n-tuple with jth entry δ_{ij} (= Kronecker delta). Then the Witt algebra W is the restricted Lie algebra $\operatorname{Der}_F \mathfrak{A} = \sum_i \mathfrak{A} D_i$, which has F-basis $\{x^{(a)} D_i \mid a \in A, 1 \leq i \leq n\}$. The bracket product in W satisfies

$$[x^{(a)} D_i, x^{(b)} D_j] = \binom{a+b-\epsilon_i}{a} x^{(a+b-\epsilon_i)} D_j - \binom{a+b-\epsilon_j}{b} x^{(a+b-\epsilon_j)} D_i,$$

and the p-mapping is p-fold composition: $D^{[p]} := D^p$ ($D \in W$). Putting $x_i = x^{(\epsilon_i)}$, we have $(x_i D_i)^{[p]} = x_i D_i$ and $(x^{(a)} D_i)^{[p]} = 0$ if $a \neq \epsilon_i$ ($1 \leq i \leq n$).

Given $a \in \mathbb{Z}^n$, set $|a| = \sum_i a_i$. Defining $\mathfrak{A}_k = \langle x^{(a)} \mid a \in A, |a| = k \rangle$ and $W_k = \sum_j \mathfrak{A}_{k+1} D_j$ we have $\mathfrak{A} = \dot{\sum}_{k=0}^{s+1} \mathfrak{A}_k$ and $W = \dot{\sum}_{k=-1}^{s} W_k$, where $s = n(p-1) - 1$. Moreover, $\mathfrak{A}_k \mathfrak{A}_l \subseteq \mathfrak{A}_{k+l}$, $[W_k, W_l] \subseteq W_{k+l}$, and $(W_k)^{[p]} \subseteq W_{pk}$ ($k, l \in \mathbb{Z}$). In particular, W_0 is a restricted Lie subalgebra of W; it is isomorphic to the general linear algebra $\mathfrak{gl}_n(F)$ via the map $x_i D_j \mapsto e_{ij}$ (= $n \times n$-matrix with 1 in the (i,j)-position and zeros elsewhere).

Let $\varphi : V \to U$ be a linear transformation of vector spaces over F. Denote by $\varphi^* : U^* \to V^*$ the dual (transpose) map given by $[\varphi^*(f)](v) = f(\varphi(v))$ ($f \in U^*$, $v \in V$). If $V = \sum_i V_i$ and $U = \sum_i U_i$ are gradings, we denote by φ_i the restriction of φ to V_i, and we say φ is *homogeneous* provided $\varphi(V_i) \subseteq U_i$ for each i.

Following Wilson [W], we use $\operatorname{Aut}^* \mathfrak{A}$ to denote the group of homogeneous automorphisms of $\mathfrak{A}$. We write $\operatorname{Aut} W$ for the group of automorphisms of W. Note that, since W is centerless, each $\Phi \in \operatorname{Aut} W$ is automatically restricted, meaning $\Phi(D^{[p]}) = \Phi(D)^{[p]}$ for all $D \in W$. Putting $\operatorname{Aut}^* W = \{\Phi \in \operatorname{Aut} W \mid \Phi \text{ is homogeneous }\}$ and $\operatorname{Aut}_1 W = \{\Phi \in \operatorname{Aut} W \mid (\Phi - 1_W)(W_i) \subseteq \sum_{j>i} W_j \text{ for each } i\}$, we have $\operatorname{Aut} W = \operatorname{Aut}^* W \ltimes \operatorname{Aut}_1 W$ [W, Theorem 2 and its following remark].

The space $\mathfrak{A}_1$ has basis $\{x_1, \dots, x_n\}$. Since $D_i x_j = \delta_{ij}$, we can regard $\{D_1, \dots, D_n\}$ as the dual basis of $\{x_1, \dots, x_n\}$ and hence identify W_{-1} with $\mathfrak{A}_1^*$. The following result is essentially an extraction of a portion of a proof in [W] which we will require.

1.1 Lemma. *The function* $\operatorname{Aut}^* W \to \mathrm{GL}((W_{-1})^*)$ *given by* $\Phi \mapsto ((\Phi_{-1})^*)^{-1}$ *is a group isomorphism.*

Proof. According to [W, Theorem 2(b, c)], we have an isomorphism $\operatorname{Aut}^* W \to \mathrm{GL}(\mathfrak{A}_1) = \mathrm{GL}((W_{-1})^*)$ given by $\Phi \mapsto \varphi_1$, where φ is the unique element of $\operatorname{Aut}^* \mathfrak{A}$ ($= \operatorname{Aut}^*(\mathfrak{A}, W)$ as defined in [W]) satisfying $\Phi(D) = \varphi D \varphi^{-1}$ for all $D \in W$. Therefore, it suffices to show that $\Phi_{-1} = (\varphi_1^{-1})^*$. For $D \in W_{-1}$, we have $\operatorname{im} D\varphi_1^{-1} \subseteq F \subseteq \mathfrak{A}$. Therefore, since φ fixes the elements of F, we have $\Phi_{-1}(D) = \varphi D \varphi_1^{-1} = D\varphi_1^{-1} = (\varphi_1^{-1})^*(D)$ for all $D \in W_{-1}$ ($= (\mathfrak{A}_1)^*$), and the proof is complete. □

The group $\operatorname{Aut} W$ acts on the set W^* according to the rule $\chi^\Phi(D) = \chi(\Phi(D))$ ($\chi \in W^*$, $\Phi \in \operatorname{Aut} W$, $D \in W$).

1.2 Theorem. *Let* $\chi \in W^*$.

(1) *If* $\Phi \in \operatorname{Aut} W$*, then* $\operatorname{ht} \chi^\Phi = \operatorname{ht} \chi$.
(2) *If* $\chi_{-1} \neq 0$*, then there exists* $\Phi \in \operatorname{Aut}^* W$ *such that* $\chi^\Phi(D_i) = \delta_{in}$ $(1 \le i \le n)$.
(3) *If* $\operatorname{ht} \chi = 1$*, then there exists* $\Phi \in \operatorname{Aut} W$ *such that* $\chi^\Phi(W_{-1}) = 0$ *and* $\chi^\Phi(x_i D_j) = 0$ *for all* $1 \le i < j \le n$.

Proof. (1) Since $\operatorname{Aut} W = \operatorname{Aut}^* W \ltimes \operatorname{Aut}_1 W$, it is enough to check the two cases $\Phi \in \operatorname{Aut}^* W$ and $\Phi \in \operatorname{Aut}_1 W$, each of which is clear.

(2) Assume $\chi_{-1} \neq 0$. Let η be the element of $(W_{-1})^*$ given by $\eta(D_i) = \delta_{in}$ $(1 \le i \le n)$. By elementary linear algebra, there exists $\psi \in \mathrm{GL}((W_{-1})^*)$ such that $\psi(\eta) = \chi_{-1}$. By 1.1, there exists $\Phi \in \operatorname{Aut}^* W$ such that $((\Phi_{-1})^*)^{-1} = \psi$. Then $(\chi^\Phi)_{-1} = (\Phi_{-1})^*(\chi_{-1}) = \psi^{-1}(\chi_{-1}) = \eta$.

(3) Assume $\operatorname{ht}(\chi) = 1$. First suppose $\chi_{-1} \neq 0$. By parts (1) and (2), we may assume $\chi(D_i) = \delta_{in}$ $(1 \le i \le n)$. Since $\chi_0 \neq 0$, we have $\chi(x_i D_j) \neq 0$ for some $1 \le i, j \le n$. We

assume i to be chosen maximal with respect to this property. Set $c = \chi(x_iD_j)^{-1}2^{-\delta_{in}}$. According to [W, Theorem 1], there exists $\Phi \in \operatorname{Aut} W$ such that $\Phi(D_k) - [cx_ix_nD_j, D_k] - D_k \in W^1$ for each k. Since $\operatorname{ht}\chi = 1$, we have $\chi(W^1) = 0$. Therefore,

$$\chi^\Phi(D_k) = \chi(\Phi(D_k)) = c\chi([x_ix_nD_j, D_k]) + \chi(D_k).$$

If $k \neq n$, then $\chi([x_ix_nD_j, D_k]) = 0$ (using maximality of i), which implies $\chi^\Phi(D_k) = 0$. Also, $[x_ix_nD_j, D_n] = -2^{\delta_{in}}x_iD_j$, so $\chi^\Phi(D_n) = 0$. Therefore, we may assume $\chi(W_{-1}) = 0$.

Let $D \in W_0$. We have $D = \sum b_{ij}x_iD_j$ and $\chi_0 = \sum c_{ij}(x_iD_j)^*$ $(b_{ij}, c_{ij} \in F)$, where $(x_iD_j)^*(x_kD_l) = \delta_{ik}\delta_{jl}$. Set $B = (b_{ij})$ and $C = (c_{ij})$. It is easy to check that $\chi_0(D) = \operatorname{tr} {}^tCB$, where ${}^tC$ denotes the transpose of the matrix C. Choose $G = (g_{ij}) \in \mathrm{GL}_n(F)$ such that GCG^{-1} is lower triangular. Let $\psi \in \mathrm{GL}(\mathfrak{A}_1)$ be given by $\psi(x_j) = \sum_i g_{ji}x_i$. As in the proof of 1.1, there exists $\Phi \in \operatorname{Aut}^* W$ such that $\varphi_1 = \psi$, where φ is the unique element of $\operatorname{Aut}^* \mathfrak{A}$ satisfying $\Phi(E) = \varphi E \varphi^{-1}$ for all $E \in W$. Computing, we have

$$\begin{aligned}\chi^\Phi(D) = \chi(\Phi(D)) = \chi(\varphi D\varphi^{-1}) = \chi_0(\varphi_1 D\varphi_1^{-1}) = \chi_0(\psi D\psi^{-1})\\ = \operatorname{tr} {}^tC({}^tGB{}^tG^{-1}) = \operatorname{tr}({}^tG^{-1}{}^tC{}^tG)B = \operatorname{tr} {}^t(GCG^{-1})B.\end{aligned}$$

Since ${}^t(GCG^{-1})$ is upper triangular, it follows that $\chi^\Phi(D) = 0$ if B is strictly upper triangular, in particular if $D = x_iD_j$ with $1 \leq i < j \leq n$. Finally, since $\Phi \in \operatorname{Aut}^* W$, we have $\chi^\Phi(W_{-1}) = \chi(W_{-1}) = 0$. □

The second part of the proof of 1.2(3) shows essentially that for $\psi \in \mathfrak{gl}_n(F)^*$, there exists $\Phi \in \operatorname{Aut}(\mathfrak{gl}_n(F))$ such that ψ^Φ vanishes on the upper triangular matrices. This was already observed in [FP, Section 1.4]. We remark also that a complete set of representatives for the character conjugacy classes in the case $W = W(1, \mathbf{1})$ was determined by Feldvoss and Nakano in [FN, 3.1].

2. Simple Induced Modules

In this section, it is shown that, with a few exceptions, the simple W-modules with character height at most one can be realized as certain induced modules $Z^\chi(S)$.

Let $\chi \in W^*$. Generalizing the construction of the restricted enveloping algebra $u(W)$ of W, one defines the *χ-reduced universal enveloping algebra* of W, denoted $u(W, \chi)$, by forming the quotient of the universal enveloping algebra $U(W)$ of W by the ideal generated by $\{D^p - D^{[p]} - \chi(D)^p 1_F \mid D \in W\}$. (Note that $u(W, 0) = u(W)$.) Just like with $u(W)$, the vector space $u(W, \chi)$ possesses a PBW-type basis. The $u(W, \chi)$-modules are precisely the W-modules having character χ. Let V be a restricted subalgebra of W. Then χ restricts to an element of V^* which we continue to denote by χ. The algebra $u(V, \chi)$ identifies with a subalgebra of $u(W, \chi)$ in the natural way. (See [SF, Section 5.3] for more details.)

There are standard restricted subalgebras of W that will be referred to throughout the paper. They are

$$N_0^- = \sum_{i>j} F x_i D_j, \qquad H = \sum_i F x_i D_i, \qquad N_0 = \sum_{i<j} F x_i D_j,$$

$$N^- = N_0^- + W_{-1}, \qquad N = N_0 + W^1, \qquad B = H + N.$$

Note that we obtain a triangular decomposition $W = N^- \dot{+} H \dot{+} N$.

Let M be a B-module. Let $\lambda \in F^n$ and set $M_\lambda = \{m \in M \mid x_i D_i \cdot m = \lambda_i m \text{ for all } 1 \leq i \leq n\}$. An element of M_λ is a *weight vector* (*of weight* λ). A nonzero $m \in M_\lambda$ is a *maximal vector* (*of weight* λ) provided $N \cdot m = 0$.

Now suppose M has character χ and let $0 \neq m \in M_\lambda$. Then $\lambda_i^p m - \lambda_i m = (x_i D_i)^p \cdot m - x_i D_i \cdot m = \chi(x_i D_i)^p m$ implying $\lambda \in \Lambda^\chi := \{\lambda \in F^n \mid \lambda_i^p - \lambda_i = \chi(x_i D_i)^p \text{ for all } 1 \leq i \leq n\}$. In particular, if M has a maximal vector of weight λ, then necessarily $\lambda \in \Lambda^\chi$. Note that if $\chi(H) = 0$, then $\Lambda^\chi = \mathbb{F}_p^n =: \Lambda$, where $\mathbb{F}_p$ is the prime subfield of F.

2.1 Lemma. *Let $\chi \in W^*$ with $\chi(N) = 0$ and let M be a $u(W, \chi)$-module. The following conditions are equivalent:*

(1) *M is nonzero and is generated by each of its maximal vectors,*
(2) *M is simple.*

Proof. Assume (1) holds and let M' be a nonzero submodule of M. Choose a simple B-submodule S of M'. Now N_0 is a p-nilpotent ideal of $H + N_0$ and the grading on W is restricted, so N is a p-nilpotent ideal of B. Since S has character χ and $\chi(N) = 0$, it follows that for each $D \in N$, we have $D^{p^l} \cdot S = D^{[p]^l} \cdot S = 0$ for some $l \in \mathbb{N}$. Therefore, $N \cdot S = 0$ [SF, Corollary 3.8, p. 19]. This implies that S is simple as H-module. Since H is abelian, S must be one-dimensional [SF, Lemma 5.6, p. 31], so $S = Fm$ for some nonzero $m \in S$. Clearly m is a maximal vector. By assumption, m generates M, so that $M' = M$. Thus (2) holds.

Since a maximal vector is nonzero by definition, the other implication is obvious. □

Since $W^1 \lhd W^0$, any W_0-module becomes a W^0-module via the canonical map $W^0 \to W^0/W^1 \cong W_0$. In particular, a W_0-module can be viewed as a B-module; the notion of maximal vector applied to this situation is the classical one (recalling that $W_0 \cong \mathfrak{gl}_n(F)$). For $\lambda \in \Lambda$, let $L_0(\lambda)$ be a restricted simple W_0-module having maximal vector of weight λ.

For any $u(W^0, \chi)$-module M, the *induced* $u(W, \chi)$-module $Z^\chi(M)$ is defined by

$$Z^\chi(M) = u(W, \chi) \otimes_{u(W^0, \chi)} M.$$

By the PBW theorem, any $v \in Z^\chi(M)$ can be expressed uniquely in the form $v = \sum_{a \in A} D^a \otimes m_a$ with $m_a \in M$, where $D^a := \prod_i D_i^{a_i}$.

If $\operatorname{ht}\chi \leq 1$ and M is a $u(W_0,\chi)$-module, then, since $\chi(W^1) = 0$, M has character χ when viewed as a W^0-module as above, so that $Z^\chi(M)$ is defined. If $\operatorname{ht}\chi \leq 0$, then $Z^\chi(L_0(\lambda))$ ($\lambda \in \Lambda$) is defined and we denote this module simply by $Z^\chi(\lambda)$.

2.2 Proposition. *Let $\chi \in W^*$ with $\operatorname{ht}\chi \leq 1$ and let M be a simple $u(W,\chi)$-module. Then M is a homomorphic image of $Z^\chi(S)$ for some simple $u(W_0,\chi)$-module S.*

Proof. M has a simple $u(W^0,\chi)$-submodule S. Now $W^1 \lhd W^0$, so arguing just as in the proof of 2.1(1), we deduce that W^1 acts trivially on S. This implies that S is a (simple) $u(W_0,\chi)$-module (see the discussion after 2.1). The inclusion map $S \to M$ is a $u(W^0,\chi)$-homomorphism, so it induces a $u(W,\chi)$-homomorphism $Z^\chi(S) \to M$, which is surjective since M is simple. $\square$

Before presenting the main result of the section, we need a technical lemma. The first part follows from [SF, Proposition 1.3(4), p. 9], and the second part is an easy consequence of the bracket product in W.

2.3 Lemma. *The following formulas hold in the algebra $u(W,\chi)$:*

(1) $(x^{(b\epsilon_i)}D_j)D_i^a = \sum_{k=0}^{b}(-1)^k\binom{a}{k}D_i^{a-k}(x^{((b-k)\epsilon_i)}D_j)$ $\quad(a, b \in \mathbb{Z}^+,\ 1 \leq i, j \leq n)$,

(2) $(x^{(\epsilon_i+\epsilon_l)}D_j)D_k = D_k(x^{(\epsilon_i+\epsilon_l)}D_j) - x^{(\epsilon_i+\epsilon_l-\epsilon_k)}D_j$ $\quad(1 \leq i, j, k, l \leq n)$. $\square$

For $0 \leq k \leq n$, set $\omega_k = -\sum_{i=k+1}^{n}\epsilon_i \in \Lambda$. The weights $\omega_0, \omega_1, \ldots, \omega_n$ are called *exceptional weights.*

Recall that $N_0^- = \sum_{i>j} Fx_iD_j$.

2.4 Theorem. *Let $\chi \in W^*$ with $\operatorname{ht}\chi \leq 1$, let M be a $u(W_0,\chi)$-module, and let v be a maximal vector in $Z^\chi(M)$ of weight λ.*

(1) *If $\chi(N_0^-) \neq 0$, then $v = 1 \otimes m$ for some maximal vector $m \in M$.*

(2) *If M has no maximal vector of weight ω_0, then either $v = 1 \otimes m$ for some maximal vector $m \in M$, or M has a maximal vector of weight ω_k for some $1 \leq k \leq n$ and we have $\lambda = \omega_{k-1}$.*

Proof. Write $v = \sum_{a\in A} D^a \otimes m_a$ with $m_a \in M$. For each $1 \leq i \leq n$, we have, using 2.3(1),

$$\sum_a D^a \otimes \lambda_i m_a = \lambda_i v = x_iD_i \cdot v = \sum_a D^a \otimes (x_iD_i \cdot m_a - a_i m_a).$$

Therefore, $x_iD_i \cdot m_a = \lambda(a)_i m_a$ for each i and a, where $\lambda(a)_i := \lambda_i + a_i$.

Step 1: *If $m_a \neq 0$, then $a_i \in \{0, 1, p-1\}$ for each i, and $\lambda(a)_i = (a_i - 1)/2$ if $a_i \neq 0$.*

First, since $N \cdot v = 0$ and $W^1 \cdot M = 0$, we get, using 2.3(1),

$$0 = x^{(2\epsilon_i)}D_i \cdot v = \sum_a \left[\binom{a_i}{2} - a_i\lambda(a)_i\right] D^{a-\epsilon_i} \otimes m_a.$$

This implies that, if $m_a \neq 0$, then either $a_i = 0$ or $\lambda(a)_i = (a_i - 1)/2$. Similarly,

$$0 = x^{(3\epsilon_i)} D_i \cdot v = \sum_a \left[-\binom{a_i}{3} + \binom{a_i}{2} \lambda(a)_i \right] D^{a-2\epsilon_i} \otimes m_a,$$

from which it follows that, if $m_a \neq 0$, then either $a_i \in \{0,1\}$ or $\lambda(a)_i = (a_i - 2)/3$. Now assume $m_a \neq 0$ and $a_i \notin \{0,1\}$ for some i. By the preceding remarks, we have $(a_i - 1)/2 = \lambda(a)_i = (a_i - 2)/3$ (in F), whence $a_i = p - 1$. This finishes step 1.

Step 2: *Assume that either* $\chi(N_0^-) \neq 0$ *or* M *has no maximal vector of weight* ω_0. *If* $m_a \neq 0$, *then* $a \in \{0\} \cup \{\epsilon_k \,|\, 1 \leq k \leq n\}$.

Fix $i \neq j$. 2.3(1) yields

$$(1) \qquad 0 = x^{(2\epsilon_i)} D_j \cdot v = -\sum_a a_i D^{a-\epsilon_i} \otimes x_i D_j \cdot m_a + \sum_a \binom{a_i}{2} D^{a-2\epsilon_i+\epsilon_j} \otimes m_a.$$

Let $a \in A$ and assume $a_i = p-1$ and $a_j \neq p-1$. Then $\binom{a_i}{2} D^{a-2\epsilon_i+\epsilon_j} \neq 0$. We contend that the term $\binom{a_i}{2} D^{a-2\epsilon_i+\epsilon_j} \otimes m_a$ in the above expression does not cancel with any other terms (thus forcing $m_a = 0$). First, if $D^{a-2\epsilon_i+\epsilon_j} = D^{b-2\epsilon_i+\epsilon_j}$ for some $b \in A$, then clearly $b = a$. Next, assume $D^{a-2\epsilon_i+\epsilon_j} = D^{b-\epsilon_i}$ for some $b \in A$. Then $a - 2\epsilon_i + \epsilon_j = b - \epsilon_i$ (since $a_j \neq p-1$), implying $a_i - 2 = b_i - 1$, or, in other words, $b_i = a_i - 1 = p - 2$. But then step 1 says $m_b = 0$ (since $p > 3$). We conclude that no cancellation occurs and hence $m_a = 0$. We have shown that if $m_a \neq 0$ and $a_i = p-1$ for some i, then $a_j = p-1$ for all j. Set $\tau = (p-1, \ldots, p-1)$. We claim that $m_\tau = 0$. Suppose instead that $m_\tau \neq 0$. For $i < j$, 2.3(1) gives

$$(2) \qquad 0 = x_i D_j \cdot v = \sum_a D^a \otimes x_i D_j \cdot m_a - \sum_a a_i D^{a-\epsilon_i+\epsilon_j} \otimes m_a.$$

Arguing as above, we conclude that $x_i D_j \cdot m_\tau = 0$ so that m_τ is a maximal vector. By step 1, m_τ has weight ω_0. Checking the assumptions of step 2, we see that it must be the case that $\chi(N_0^-) \neq 0$, so that $\chi(x_i D_j) \neq 0$ for some $i > j$. Since $(x_i D_j)^p \cdot x = \chi(x_i D_j)^p x$ ($x \in M$), it follows that $x_i D_j \cdot x \neq 0$ for each nonzero $x \in M$. In particular, $x_i D_j \cdot m_\tau \neq 0$. Therefore, the term $D^{\tau - \epsilon_i} \otimes x_i D_j \cdot m_\tau$ in equation (1) is nonzero, and it is easily seen that it does not cancel with any other terms. This is a contradiction. Therefore, the claim that $m_\tau = 0$ is established.

So far, we have shown that if $m_a \neq 0$, then $a_i \in \{0,1\}$ for each i. Let $a \in A$ and assume $a_i = 1 = a_j$ for some $i < j$. Then $a_i D^{a-\epsilon_i+\epsilon_j} \neq 0$ and the term $a_i D^{a-\epsilon_i+\epsilon_j} \otimes m_a$ in equation (2) does not cancel with any other terms (for if $D^{a-\epsilon_i+\epsilon_j} = D^b$ for some $b \in A$, then $a - \epsilon_i + \epsilon_j = b$, implying $b_j = a_j + 1 = 2$, whence $m_b = 0$). Thus $m_a = 0$.

Summarizing, if $m_a \neq 0$, then $a_i \in \{0, 1\}$ for each i, and $a_i = 1$ for at most one i. This completes step 2.

Step 3: *Completion of proof.*

(1) Assume $\chi(N_0^-) \neq 0$ so that $\chi(x_i D_j) \neq 0$ for some $i > j$. By step 2, we have $v = 1 \otimes m_0 + \sum_k D_k \otimes m_{\epsilon_k}$. Equation (1) gives $1 \otimes x_i D_j \cdot m_{\epsilon_i} = 0$, which implies $m_{\epsilon_i} = 0$ (using the argument in the proof of step 2). Then using 2.3(2), we get, for each k, $-1 \otimes x_i D_j \cdot m_{\epsilon_k} = x^{(\epsilon_i + \epsilon_k)} D_j \cdot v = 0$, implying $m_{\epsilon_k} = 0$. Hence, $v = 1 \otimes m_0$ and, since v is a maximal vector, so is m_0.

(2) Assume M has no maximal vector of weight ω_0. Again, step 2 applies to give $v = 1 \otimes m_0 + \sum_k D_k \otimes m_{\epsilon_k}$. If $m_{\epsilon_k} = 0$ for all $1 \leq k \leq n$, then $v = 1 \otimes m_0$ and m_0 is a maximal vector. So assume $m_{\epsilon_l} \neq 0$ for some $1 \leq l \leq n$. Assume further that l is the least such integer. We shall show that

$$\lambda(\epsilon_l)_i = \begin{cases} 0, & i \leq l, \\ -1, & i > l. \end{cases} \tag{3}$$

If $i \leq l$, then $0 = x^{(\epsilon_i + \epsilon_l)} D_i \cdot v = -\lambda(\epsilon_l)_i 1 \otimes m_{\epsilon_l}$ by 2.3(2), so $\lambda(\epsilon_l)_i = 0$. Next, we note that, for $j < i$, 2.3(1) gives

$$0 = x_j D_i \cdot v = 1 \otimes x_j D_i \cdot m_0 + \sum_{k \geq l} D_k \otimes x_j D_i \cdot m_{\epsilon_k} - \begin{cases} 0, & j < l, \\ D_i \otimes m_{\epsilon_j}, & j \geq l. \end{cases} \tag{4}$$

Let $i > l$. Setting $j = l$ in equation (4) we find that $x_l D_i \cdot m_{\epsilon_i} = m_{\epsilon_l}$. In particular, $m_{\epsilon_i} \neq 0$, so step 1 says $\lambda(\epsilon_i)_i = 0$, that is, $x_i D_i \cdot m_{\epsilon_i} = 0$. Therefore, $x_i D_i \cdot m_{\epsilon_l} = x_i D_i \cdot (x_l D_i \cdot m_{\epsilon_i}) = -x_l D_i \cdot m_{\epsilon_i} = -m_{\epsilon_l}$, implying $\lambda(\epsilon_l)_i = -1$. This establishes equation (3). Equation (4) also shows that $x_j D_i \cdot m_{\epsilon_l} = 0$ whenever $j < i$, implying m_{ϵ_l} is a maximal vector. By equation (3) its weight is ω_l. Finally, $D_l \otimes m_{\epsilon_l}$ has weight ω_{l-1}, which implies that the weight λ of v is ω_{l-1} as well. □

Let $\mathfrak{g}$ be a restricted Lie algebra. Let $\Phi \in \mathrm{Aut}(\mathfrak{g})$ and let M be a $\mathfrak{g}$-module. Denote by M^Φ the $\mathfrak{g}$-module having M as its underlying vector space and $\mathfrak{g}$-action given by $x \cdot m = \Phi(x)m$ ($x \in \mathfrak{g}$, $m \in M$), where the action on the right is the given one. Clearly, M^Φ is simple if and only if M is. Also, it is easy to check that if M has character χ, then M^Φ has character χ^Φ.

From the description before 1.1 of $\mathrm{Aut}\, W$, we see that any $\Phi \in \mathrm{Aut}\, W$ restricts to an automorphism of W^0 (resp., W^1), which we continue to denote by Φ. The next proposition will allow us to reduce certain arguments to the situation of 2.1.

2.5 Proposition. *Let $\chi \in W^*$ and let $\Phi \in \mathrm{Aut}\, W$.*

(1) *If M is a $u(W^0, \chi)$-module, then $[Z^\chi(M)]^\Phi \cong Z^{\chi^\Phi}(M^\Phi)$.*
(2) *If $\mathrm{ht}\, \chi \leq 0$, then $[Z^\chi(\omega_k)]^\Phi \cong Z^{\chi^\Phi}(\omega_k)$ $(0 \leq k \leq n)$.*

Proof. (1) Let M be a $u(W^0, \chi)$-module. As noted above, $[Z^\chi(M)]^\Phi$ is a $u(W, \chi^\Phi)$-module. Its subspace $1 \otimes M$ is a $u(W^0, \chi^\Phi)$-submodule isomorphic to M^Φ. Moreover, a $u(W^0, \chi^\Phi)$-isomorphism $M^\Phi \to 1 \otimes M$ induces a $u(W, \chi^\Phi)$-homomorphism $\varphi : Z^{\chi^\Phi}(M^\Phi) \to [Z^\chi(M)]^\Phi$, which is necessarily surjective since $1 \otimes M$ generates $[Z^\chi(M)]^\Phi$. Finally, both modules have dimension $p^n \dim_F M$, so φ is an isomorphism.

(2) First, we describe the usual concrete realization of the W_0-module $L_0(\omega_k)$. Recall the isomorphism $W_0 \cong \mathfrak{gl}_n(F) =: \mathfrak{g}$. Let V be the natural module for $\mathfrak{g}$. It is well known that $V_k := \bigwedge_{i=1}^k V$ is a simple restricted $\mathfrak{g}$-module with maximal vector of weight $(1, \dots, \underset{k}{1}, 0, \dots, 0)$. ($V$ consists of n-dimensional column vectors. If v_i is the column vector with 1 in the ith position and 0's elsewhere, then $v_1 \wedge \cdots \wedge v_k$ is clearly a maximal vector of the indicated weight in V_k.) Let T denote the one-dimensional trace module for $\mathfrak{g}$ ($T = Fv$ with $xv = (\operatorname{tr} x)v$ for $x \in \mathfrak{g}$). Then $L_0(\omega_k) \cong T^{p-1} \otimes_F V_k$, where $T^{p-1} = T \otimes_F \cdots \otimes_F T$ ($p-1$ factors). Indeed, $T^{p-1} \otimes_F V_k$ is clearly simple (since V_k is) and it contains $v^{p-1} \otimes (v_1 \wedge \cdots \wedge v_k)$, a maximal vector of weight $(p-1, \dots, p-1) + (1, \dots, \underset{k}{1}, 0, \dots, 0) = (0, \dots, \underset{k}{0}, -1, \dots, -1) = \omega_k$.

Assume $\operatorname{ht} \chi \leq 0$. By (1), it suffices to show that $L_0(\omega_k)^\Phi$ and $L_0(\omega_k)$ are isomorphic as modules for $u(W^0, \chi) = u(W^0)$. Because W^1 acts trivially on $L_0(\omega_k)$, we may assume $\Phi \in \operatorname{Aut}^* W$. Let $\varphi \in \operatorname{Aut}^* \mathfrak{A}$ be as in the proof of 1.1, so that $\Phi(D) = \varphi D \varphi^{-1}$ for all $D \in W$. We view φ_1 as an element of $\operatorname{GL}(V)$ by identifying the vector space V with $\mathfrak{A}_1$ via $v_i \mapsto x_i$. Then, with the usual diagonal action of $\operatorname{GL}(V)$ on V_k, we have that $v^{p-1} \otimes \varphi_1 \cdot (v_1 \wedge \cdots \wedge v_k)$ is a maximal vector in $(T^{p-1} \otimes_F V_k)^\Phi \cong L_0(\omega_k)^\Phi$ of weight ω_k. Since $L_0(\omega_k)^\Phi$ is simple, we have $L_0(\omega_k)^\Phi \cong L_0(\omega_k)$ as desired. □

2.6 Corollary. *Let $\chi \in W^*$ with $\operatorname{ht} \chi \leq 1$, and let S be a simple $u(W_0, \chi)$-module. If S is not W_0-isomorphic to any $L_0(\omega_k)$ $(0 \leq k \leq n)$, then $Z^\chi(S)$ is simple. In particular, if $\operatorname{ht} \chi = 1$, then $Z^\chi(S)$ is simple.*

Proof. If $\operatorname{ht} \chi = 1$, then $\chi(W_0) \neq 0$, implying S is not restricted and hence not W_0-isomorphic to any $L_0(\omega_k)$ $(0 \leq k \leq n)$. Therefore, it suffices to prove the first statement.

In view of 2.5 (and its proof), we may replace χ with any convenient conjugate χ^Φ, $\Phi \in \operatorname{Aut} W$. Then by 1.2 we may assume $\chi(N) = 0$.

Assume S is not W_0-isomorphic to any $L_0(\omega_k)$ $(0 \leq k \leq n)$. Let $v \in Z^\chi(S)$ be a maximal vector of weight, say, λ. By 2.1, it is enough to show that v generates $Z^\chi(S)$.

First assume $\chi(N_0^-) \neq 0$. By 2.4(1), $v = 1 \otimes m$ with $0 \neq m \in S$. Since m generates S, it follows that v generates $Z^\chi(S)$.

For the remainder of the proof, assume $\chi(N_0^-) = 0$. Suppose $\operatorname{ht} \chi \leq 0$. Then S is restricted and hence, by assumption, does not contain a maximal vector of exceptional weight ω_k for any k. Therefore, 2.4(2) says v is of the form $1 \otimes m$ with $0 \neq m \in S$, and v generates $Z^\chi(S)$ as before.

Finally suppose $\operatorname{ht}\chi = 1$. Then $\chi(x_iD_i) \neq 0$ for some i. As pointed out before 2.1, $\lambda \in \Lambda^\chi$, so λ_i is a solution of $\lambda_i^p - \lambda_i = \chi(x_iD_i)^p$. In particular, $\lambda_i \notin \{0, 1\}$. This shows that λ is not exceptional. Once again, 2.4(2) says v is of the form $1 \otimes m$ with $0 \neq m \in S$ and the proof is complete. □

3. Exceptional Simple Modules

Here, we study the structure of those induced modules that were not included in 2.6, namely, the modules $Z^\chi(\omega_k)$ with $0 \leq k \leq n$, $\operatorname{ht}\chi \leq 0$. When $\operatorname{ht}\chi = -1$ (the restricted case), these modules are isomorphic to the terms in the usual de Rham complex for W and the properties of the complex have been used to determine the structure of the induced modules (see [Sh], [N]). In order to handle the case $\operatorname{ht}\chi = 0$, we define a more general de Rham complex–one depending on χ.

For this entire section, we fix $\chi \in W^*$ and assume $\operatorname{ht}\chi \leq 0$. Set

$$\mathfrak{A}^\chi = \operatorname{Hom}_{u(W^0)}(u(W, \chi), F),$$

where F is the trivial W^0-module. For any n-tuple a of nonnegative integers, put $D^a = \prod_i D_i^{a_i}$. Since $u(W, \chi)$ is a free left $u(W^0)$-module with base $\{D^a \mid a \in A\}$ (by the PBW theorem), we get an F-basis $\{y^{(a)} \mid a \in A\}$ of $\mathfrak{A}^\chi$ by defining $y^{(a)}(vD^b) = \epsilon(v)\delta_{ab}$, where $v \in u(W^0)$ and $\epsilon : u(W^0) \to F$ is the augmentation homomorphism, which satisfies $\epsilon(D) = 0$ for all $D \in W^0$.

For any $b \in \mathbb{Z}^n$, there exist unique $b^0 \in A$, $b^1 \in (p\mathbb{Z})^n$ such that $b = b^0 + b^1$. We extend the notation of the previous paragraph by defining for $b \in \mathbb{Z}^n$

$$y^{(b)} = \begin{cases} \chi(D)^{-b^1} y^{(b^0)}, & b \leq \tau, \\ 0, & \text{otherwise}, \end{cases}$$

where $\chi(D)^c := \prod_i \chi(D_i)^{c_i}$ for $c \in \mathbb{Z}^n$, $c \geq 0$ (with the convention $0^0 = 1$), and where $\tau = (p-1, \ldots, p-1)$ as before. From the definition of $u(W, \chi)$, we have $D^c = \chi(D)^{c^1} D^{c^0}$ ($c \in \mathbb{Z}^n$, $c \geq 0$), and therefore,

$$y^{(b)}(vD^c) = \epsilon(v)\chi(D)^{c^1 - b^1} \delta_{b^0, c^0}$$

($v \in u(W^0)$, $b, c \in \mathbb{Z}^n$, $b \leq \tau$, $c \geq 0$).

We extend the definition of the binomial coefficient $\binom{l}{k}$ to arbitrary $l, k \in \mathbb{Z}$ in the usual way:

$$\binom{l}{k} = \begin{cases} \frac{l(l-1)\cdots(l-k+1)}{k(k-1)\cdots 1}, & k > 0, \\ 1, & k = 0, \\ 0, & k < 0 \end{cases}$$

(and put $\binom{a}{b} = \prod_i \binom{a_i}{b_i}$ for $a, b \in \mathbb{Z}^n$). The standard identity

$$\binom{l+1}{k} = \binom{l}{k} + \binom{l}{k-1}$$

is easily seen to be valid for all $l, k \in \mathbb{Z}$. Using the easily checked identity $\binom{l}{k} = (-1)^k \binom{-l+k-1}{k}$ for the case $l < 0$, we see that $\binom{l}{k}$ is always an integer. The proof of the following lemma is straightforward.

3.1 Lemma. *Let $k, l, m \in \mathbb{Z}$.*

(1) *If $0 \le k < p$, $l < p$, and $k + l \ge p$, then $\binom{k+l}{k} \equiv 0 \pmod{p}$.*
(2) *If $0 \le k < p$ and $l \equiv m \pmod{p}$, then $\binom{l}{k} \equiv \binom{m}{k} \pmod{p}$.* □

Let $\Delta : U(W) \to U(W) \otimes_F U(W)$ be the usual comultiplication, which is induced by $D \mapsto D \otimes 1 + 1 \otimes D$ $(D \in W)$. The composition $U(W) \xrightarrow{\Delta} U(W) \otimes_F U(W) \xrightarrow{\text{can.}} u(W) \otimes_F u(W, \chi)$ sends $D^p - D^{[p]} - \chi(D)^p$ $(D \in W)$ to zero and hence induces a map $\Delta' : u(W, \chi) \to u(W) \otimes_F u(W, \chi)$ which makes $u(W, \chi)$ into a left $u(W)$-comodule.

For $x \in \mathfrak{A}^0$, $y \in \mathfrak{A}^\chi$, define $xy : u(W, \chi) \to F$ by $(xy)(u) = x\overline{\otimes}y(\Delta'(u))$. (Here, $x\overline{\otimes}y(u_1 \otimes u_2) := x(u_1)y(u_2)$.) One easily checks using [SF, p. 125, last paragraph], that $xy \in \mathfrak{A}^\chi$. In the case $\chi = 0$, this gives an associative product on $\mathfrak{A}^0$, the associativity coming from the coassociativity of Δ, and thus $\mathfrak{A}^0$ becomes an associative algebra. In turn, for general χ the above product makes $\mathfrak{A}^\chi$ into a left $\mathfrak{A}^0$-module. Similarly, $\mathfrak{A}^\chi$ becomes a right $\mathfrak{A}^0$-module.

If $\chi = 0$, then $y^{(b)} = 0$ for $b \notin A$. (Indeed, if $b \not\le \tau$, then $y^{(b)} = 0$ by definition, and if $b \le \tau$ and $b \not\ge 0$, then $b^1 \ne 0$, so $y^{(b)} = 0(D)^{-b^1} y^{(b^0)} = 0$.) Therefore, since $x^{(b)} = 0$ for $b \notin A$ as well, we obtain a well-defined vector space isomorphism $\mathfrak{A}^0 \to \mathfrak{A}$ via $y^{(b)} \mapsto x^{(b)}$ $(b \in \mathbb{Z}^n)$. We use this isomorphism to identify these two spaces. It follows from the next result that this is actually an algebra isomorphism. (We retain the notation $y^{(b)}$ for the aforementioned basis vector of $\mathfrak{A}^\chi$ with χ our arbitrary fixed character.)

3.2 Lemma. *If $a, b \in \mathbb{Z}^n$ and $a \le \tau$, then $x^{(a)} y^{(b)} = \binom{a+b}{a} y^{(a+b)}$.*

Proof. Let $a, b \in \mathbb{Z}^n$ and assume $a \le \tau$. We first note that if $a, b \in A$, then the formula can be derived just as in the proof of [SF, Lemma 5.7(2), p. 131]. We will use this observation to establish the general case below.

If $a \not\ge 0$, then $x^{(a)} = 0$, and also $\binom{a+b}{a} = 0$ by the definition of the binomial coefficient, so both sides of the equation equal zero if $a \notin A$. Therefore, we may assume $a \in A$. If $b \not\le \tau$, then $a + b \not\le \tau$, and so $y^{(b)} = 0 = y^{(a+b)}$, giving the lemma. Hence, we may assume $b \le \tau$. From the previous paragraph, we get $x^{(a)} y^{(b)} = \chi(D)^{-b^1} x^{(a)} y^{(b^0)} = \chi(D)^{-b^1} \binom{a+b^0}{a} y^{(a+b^0)}$. If $a + b^0 \notin A$, then $\binom{a+b}{a} = \binom{a+b^0}{a} = 0$ (in F), using 3.1(2) then 3.1(1), so the lemma holds. On the other hand, if $a+b^0 \in A$, then $(a+b)^0 = a+b^0$ and $(a+b)^1 = b^1$, implying $x^{(a)}y^{(b)} = \binom{a+b^0}{a}\chi(D)^{-(a+b)^1}y^{((a+b)^0)} = \binom{a+b}{a}y^{(a+b)}$. □

For reference, we state a generalization of the standard binomial coefficient identity stated before 3.1; the proof is straightforward.

3.3 Lemma. $\binom{a+\epsilon_i}{b} = \binom{a}{b} + \binom{a}{b-\epsilon_i}$ $(a, b \in \mathbb{Z}^n,\ 1 \le i \le n)$. □

$\mathfrak{A}^\chi$ is a coinduced $u(W,\chi)$-module; the W-action is given by $(D\cdot y)(u) = y(uD)$ $(D \in W$, $y \in \mathfrak{A}^\chi$, $u \in u(W,\chi))$. (The next result shows that when $\chi = 0$ this action coincides with the usual action of W on the divided power algebra $\mathfrak{A}$.)

3.4 Proposition. *If $1 \le i \le n$, $a \in A$, and $b \in \mathbb{Z}^n$ with $b \le \tau$, then $x^{(a)}D_i \cdot y^{(b)} = \binom{a+b-\epsilon_i}{a}y^{(a+b-\epsilon_i)}$.*

Proof. Assume the hypotheses. One checks by induction on $c \in A$ that in the algebra $u(W,\chi)$ we have the formula

$$D^c(x^{(a)}D_i) = \sum_{0\le e\le a\wedge c}\binom{c}{e}(x^{(a-e)}D_i)D^{c-e},$$

where $(a\wedge c) \in A$ satisfies $(a\wedge c)_j = \min\{a_j, c_j\}$. Then, for $v \in u(W^0)$, $c \in A$, we have

$$\begin{aligned}
(x^{(a)}D_i \cdot y^{(b)})(vD^c) = y^{(b)}(vD^c(x^{(a)}D_i)) &= \chi(D)^{-b^1}\sum_{0\le e\le a\wedge c}\binom{c}{e}y^{(b^0)}(v(x^{(a-e)}D_i)D^{c-e}) \\
&= \binom{c}{a}\chi(D)^{(c-a+\epsilon_i)^1-b^1}\epsilon(v)\delta_{b^0,(c-a+\epsilon_i)^0} \\
&= \binom{a+b-\epsilon_i}{a}\chi(D)^{c^1-(a+b-\epsilon_i)^1}\epsilon(v)\delta_{(a+b-\epsilon_i)^0,c^0} \\
&= \binom{a+b-\epsilon_i}{a}y^{(a+b-\epsilon_i)}(vD^c),
\end{aligned}$$

where, for the third equality we have used that $x^{(a-e)}D_i \in W^0$ unless $e = a$, and for the fourth equality we have used that $r^0 = s^0$ implies $r^1 - s^1 = r - s$ $(r, s \in \mathbb{Z}^n)$, as well as 3.1(2). The lemma follows. □

3.5 Corollary. *Let $x \in \mathfrak{A}$, $y \in \mathfrak{A}^\chi$, $D \in W$.*

(1) $(xD)\cdot y = x(D\cdot y)$,
(2) $D\cdot(xy) = (Dx)y + x(D\cdot y)$.

Proof. We may assume $x = x^{(a)}$, $y = y^{(b)}$, $D = x^{(c)}D_j$, with $a, b, c \in A$, $1 \le j \le n$.

(1) We have

$$(xD)\cdot y = \binom{a+c}{a}x^{(a+c)}D_j\cdot y^{(b)} = \binom{a+c}{a}\binom{a+c+b-\epsilon_j}{a+c}y^{(a+c+b-\epsilon_j)},$$

using 3.4 if $a + c \leq \tau$, and the fact that $\binom{a+c}{a} 1_F = 0$ if $a + c \not\leq \tau$ (3.1(1)). On the other hand,

$$x(D \cdot y) = \binom{c+b-\epsilon_j}{c} x^{(a)} y^{(c+b-\epsilon_j)} = \binom{c+b-\epsilon_j}{c} \binom{a+c+b-\epsilon_j}{a} y^{(a+c+b-\epsilon_j)},$$

using 3.4, then 3.2. The desired equality now follows easily from the definition of the binomial coefficient.

(2) Using (1), we reduce to the case $D = D_j$, with $1 \leq j \leq n$. Then

$$D \cdot (xy) = \binom{a+b}{a} D_j \cdot y^{(a+b)} = \binom{a+b}{a} y^{(a+b-\epsilon_j)},$$

while

$$(Dx)y + x(D \cdot y) = x^{(a-\epsilon_j)} y^{(b)} + x^{(a)} y^{(b-\epsilon_j)} = \left[\binom{a-\epsilon_j+b}{a-\epsilon_j} + \binom{a+b-\epsilon_j}{a} \right] y^{(a+b-\epsilon_j)}.$$

The desired equality now follows from 3.3. □

We continue to view W and $\mathfrak{A}$ as left $\mathfrak{A}$-modules in the natural way and set $\Omega_1 = \mathrm{Hom}_{\mathfrak{A}}(W, \mathfrak{A})$. Then Ω_1 is a free $\mathfrak{A}$-module with base $\{dx_1, \ldots, dx_n\}$, where $d : \mathfrak{A} \to \Omega_1$ is given by $dx : D \mapsto Dx$ ($x \in \mathfrak{A}$, $D \in W$). Ω_1 becomes a (restricted) W-module by defining $D \cdot \varphi = D \circ \varphi - \varphi \circ (\mathrm{ad}\, D)$ ($D \in W$, $\varphi \in \Omega_1$) (cf. [BW, p. 125]). The following formula is routine to verify.

3.6 Lemma. $x^{(a)} D_i \cdot dx_j = \delta_{ij} \sum_{k=1}^{n} x^{(a-\epsilon_k)} dx_k$ ($a \in A$, $1 \leq i, j \leq n$). □

The exterior algebra Ω of Ω_1 over $\mathfrak{A}$ is a W-module with W-action extended from the actions on Ω_1 and $\mathfrak{A}$ subject to the rules $D \cdot (v \wedge w) = (D \cdot v) \wedge w + v \wedge (D \cdot w)$, $D \cdot (xv) = (Dx)v + x(D \cdot v)$ ($D \in W$, $x \in \mathfrak{A}$, and $v, w \in \Omega$). We have $\Omega = \dot{\sum}_{k=0}^{n} \Omega_k$, where Ω_k is the k-fold exterior power of Ω_1 over $\mathfrak{A}$. Ω_k is a W-submodule of Ω; it has F-basis $\{x^{(a)} dx_\gamma \mid a \in A, \gamma \in \Gamma_k\}$, where $\Gamma_k := \{\gamma \in \mathbb{Z}^k \mid 1 \leq \gamma_1 < \gamma_2 < \cdots < \gamma_k \leq n\}$ and $dx_\gamma := dx_{\gamma_1} \wedge \cdots \wedge dx_{\gamma_k}$.

Set $\Omega_k^\chi = \mathfrak{A}^\chi \otimes_{\mathfrak{A}} \Omega_k$. By the right-hand analog of 3.2, $\mathfrak{A}^\chi$ is a free right $\mathfrak{A}$-module with base $\{y^{(0)}\}$. (In fact, 3.2 shows that $\mathfrak{A}^\chi$ is isomorphic to $\mathfrak{A}$ as left (resp., right) $\mathfrak{A}$-module.) Therefore, Ω_k^χ is a free left $\mathfrak{A}$-module with base $\{y^{(0)} \otimes dx_\gamma \mid \gamma \in \Gamma_k\}$ and hence it has F-basis $\{y^{(a)} \otimes dx_\gamma \mid a \in A, \gamma \in \Gamma_k\}$.

The tensor product $\mathfrak{A}^\chi \otimes_F \Omega_k$ becomes a W-module in the usual way: $D \cdot (y \otimes v) = (D \cdot y) \otimes v + y \otimes (D \cdot v)$ ($D \in W$, $y \in \mathfrak{A}^\chi$, $v \in \Omega_k$). By using 3.5(2), one checks that this action induces a well-defined W-module structure on $\mathfrak{A}^\chi \otimes_{\mathfrak{A}} \Omega_k = \Omega_k^\chi$. Clearly, Ω_k^χ has character χ. Putting the above action together with 3.6 gives the formula in the following proposition.

3.7 Proposition. *If* $1 \le i, k \le n$, $\gamma \in \Gamma_k$, $a \in A$, *and* $b \in \mathbb{Z}^n$ *with* $b \le \tau$, *then*

$$x^{(a)} D_i \cdot (y^{(b)} \otimes dx_\gamma) = \binom{a+b-\epsilon_i}{a} y^{(a+b-\epsilon_i)} \otimes dx_\gamma + \delta_{i\in\gamma} \sum_{j \notin \gamma\setminus i} (-1)^{|{}_i\gamma_j|} \binom{a-\epsilon_j+b}{a-\epsilon_j} y^{(a-\epsilon_j+b)} \otimes dx_{(\gamma\setminus i)\cup j}. \qquad \square$$

In this proposition and below, if P is a statement, then we define δ_P to be 1 if P is true and 0 otherwise. Also, we abuse notation slightly by viewing the k-tuple γ as a set. Then ${}_i\gamma_j = \{l \in \gamma \,|\, l \text{ is between } i \text{ and } j\}$ and $(\gamma\setminus i)\cup j$ is the k-tuple obtained by replacing i with j and reordering; it is the reordering that accounts for the sign $(-1)^{|{}_i\gamma_j|}$ in the proposition.

Recall that $Z^\chi(\lambda) := u(W,\chi) \otimes_{u(W^0)} L_0(\lambda)$, where $L_0(\lambda)$ is a simple restricted W_0-module having maximal vector of weight λ.

3.8 Proposition. $\Omega_k^\chi \cong Z^\chi(\omega_k)$ $(0 \le k \le n)$.

Proof. Let the notation be as in the proof of 2.5(2). It is not hard to check that $M := \langle y^{(\tau)} \otimes dx_\gamma \,|\, \gamma \in \Gamma_k\rangle \le \Omega_k^\chi$ (where $\tau = (p-1,\dots,p-1)$) is W^0-isomorphic to $T^{p-1} \otimes_F V_k$ (viewed as a W^0-module via $W^0 \twoheadrightarrow W^0/W^1 \cong W_0 \cong \mathfrak{g}$). An isomorphism is given by $y^{(\tau)} \otimes dx_\gamma \mapsto v^{p-1} \otimes v_\gamma$. Therefore, $M \cong L_0(\omega_k)$.

Since, as noted above, Ω_k^χ is a $u(W,\chi)$-module, an isomorphism $L_0(\omega_k) \to M$ induces a $u(W,\chi)$-homomorphism $\varphi : Z^\chi(\omega_k) \to \Omega_k^\chi$, the image of which contains M. Now $y^{(a)} \otimes dx_\gamma = D^{\tau-a} \cdot (y^{(\tau)} \otimes dx_\gamma) \in \operatorname{im}\varphi$ $(a \in A, \gamma \in \Gamma_k)$, so φ is surjective. Since both modules have dimension $p^n\binom{n}{k}$, φ is an isomorphism. $\square$

Define a linear map $\delta_k^\chi : \Omega_k^\chi \to \Omega_{k+1}^\chi$ by

$$\delta_k^\chi(y^{(b)} \otimes dx_\gamma) = \sum_{i=1}^n (D_i \cdot y^{(b)}) \otimes dx_\gamma \wedge dx_i = \sum_{i\notin\gamma} (-1)^{|\gamma_{>i}|} y^{(b-\epsilon_i)} \otimes dx_{\gamma\cup i}$$

$(b \in A, \gamma \in \Gamma_k)$, where $\gamma_{>i} = \{j \in \gamma \,|\, j > i\}$. One easily checks that the same formula is valid for any $b \in \mathbb{Z}^n$ with $b \le \tau$.

3.9 Theorem. δ_k^χ *is a* W*-homomorphism.*

Proof. Let $a, b \in A$, $1 \le i \le n$, $\gamma \in \Gamma_k$. It is enough to show that $\delta_k^\chi(x^{(a)}D_i \cdot (y^{(b)} \otimes dx_\gamma)) = x^{(a)}D_i \cdot \delta_k^\chi(y^{(b)} \otimes dx_\gamma)$. Using the definitions and 3.4, we get

$$\delta_k^\chi(x^{(a)}D_i \cdot (y^{(b)} \otimes dx_\gamma)) = S_1 + S,$$

where

$$S_1 = \binom{a+b-\epsilon_i}{a} \sum_{l\notin\gamma} (-1)^{|\gamma_{>l}|} y^{(a+b-\epsilon_i-\epsilon_l)} \otimes dx_{\gamma\cup l},$$

$$S = \delta_{i\in\gamma} \sum_{j\notin\gamma\setminus i} (-1)^{|{}_i\gamma_j|} \binom{a-\epsilon_j+b}{a-\epsilon_j} \sum_{l \notin (\gamma\setminus i)\cup j} (-1)^{|(\gamma\setminus i)\cup j_{>l}|} y^{(a-\epsilon_j+b-\epsilon_l)} \otimes dx_{(\gamma\setminus i)\cup j\cup l}.$$

(In computing S_1, we have used the formula for δ_k^χ to compute $\delta_k^\chi(y^{(a+b-\epsilon_i)} \otimes dx_\gamma)$. Actually, the formula might not apply if $a+b-\epsilon_i \not\leq \tau$. However, in this case, $y^{(a+b-\epsilon_i)} = 0$ and $\binom{a+b-\epsilon_i}{a} = 0$ (3.1(1)), so the expression for S_1 still holds. A similar statement applies to the sum S.) Separating out the terms in S with $j = i$ and $l = i$ gives $S = S_2 + S_3 + S_4$, where

$$S_2 = \delta_{i\in\gamma} \sum_{l\notin\gamma} (-1)^{|\gamma_{>l}|} \binom{a-\epsilon_i+b}{a-\epsilon_i} y^{(a-\epsilon_i+b-\epsilon_l)} \otimes dx_{\gamma\cup l},$$

$$S_3 = \delta_{i\in\gamma} \sum_{j\notin\gamma} (-1)^{|{}_i\gamma_j|} \binom{a-\epsilon_j+b}{a-\epsilon_j} (-1)^{|\gamma\cup j_{>i}|} y^{(a-\epsilon_j+b-\epsilon_i)} \otimes dx_{\gamma\cup j},$$

$$S_4 = \delta_{i\in\gamma} \sum_{j\notin\gamma} (-1)^{|{}_i\gamma_j|} \binom{a-\epsilon_j+b}{a-\epsilon_j} \sum_{l\notin\gamma\cup j} (-1)^{|(\gamma\setminus i)\cup j_{>l}|} y^{(a-\epsilon_j+b-\epsilon_l)} \otimes dx_{(\gamma\setminus i)\cup j\cup l}.$$

Similarly, we get

$$x^{(a)} D_i \cdot \delta_k^\chi(y^{(b)} \otimes dx_\gamma) = T_1 + T,$$

where

$$T_1 = \sum_{l\notin\gamma} (-1)^{|\gamma_{>l}|} \binom{a+b-\epsilon_l-\epsilon_i}{a} y^{(a+b-\epsilon_l-\epsilon_i)} \otimes dx_{\gamma\cup l},$$

$$T = \sum_{l\notin\gamma} (-1)^{|\gamma_{>l}|} \delta_{i\in\gamma\cup l} \sum_{j\notin(\gamma\cup l)\setminus i} (-1)^{|{}_i(\gamma\cup l)_j|} \binom{a-\epsilon_j+b-\epsilon_l}{a-\epsilon_j} y^{(a-\epsilon_j+b-\epsilon_l)} \otimes dx_{((\gamma\cup l)\setminus i)\cup j}.$$

Separating out the terms in T with $l = i$ and $j = i$ gives $T = T_2 + T_3 + T_4$, where

$$T_2 = \delta_{i\notin\gamma} \sum_{j\notin\gamma} (-1)^{|\gamma_{>i}|} (-1)^{|{}_i\gamma_j|} \binom{a-\epsilon_j+b-\epsilon_i}{a-\epsilon_j} y^{(a-\epsilon_j+b-\epsilon_i)} \otimes dx_{\gamma\cup j},$$

$$T_3 = \delta_{i\in\gamma} \sum_{l\notin\gamma} (-1)^{|\gamma_{>l}|} \binom{a-\epsilon_i+b-\epsilon_l}{a-\epsilon_i} y^{(a-\epsilon_i+b-\epsilon_l)} \otimes dx_{\gamma\cup l},$$

$$T_4 = \delta_{i\in\gamma} \sum_{\substack{j,l\notin\gamma \\ j\neq l}} (-1)^{|\gamma_{>l}|} (-1)^{|{}_i(\gamma\cup l)_j|} \binom{a-\epsilon_j+b-\epsilon_l}{a-\epsilon_j} y^{(a-\epsilon_j+b-\epsilon_l)} \otimes dx_{(\gamma\setminus i)\cup j\cup l}.$$

Therefore, it remains to verify that $S_1 + S_2 + S_3 + S_4 = T_1 + T_2 + T_3 + T_4$.

If $i \notin \gamma$, then $S_2 = S_3 = S_4 = T_3 = T_4 = 0$ and $T_1 + T_2 = S_1$ (using 3.3), so this case is checked. Now assume $i \in \gamma$. One easily checks (using 3.3 again), that

$$S_1 + S_2 = \sum_{l\notin\gamma} (-1)^{|\gamma_{>l}|} \binom{a+b}{a} y^{(a+b-\epsilon_i-\epsilon_l)} \otimes dx_{\gamma\cup l} = T_1 + T_3 - S_3.$$

Next,

$$S_4 - T_4 = \sum_{\substack{j,l \notin \gamma \\ j \neq l}} (-1)^{\delta_{l<j}} (-1)^{|\gamma_{>i}|} (-1)^{|_j(\gamma \setminus i)_l|} \binom{a+b-\epsilon_j-\epsilon_l}{a-\epsilon_j-\epsilon_l} y^{(a+b-\epsilon_j-\epsilon_l)} \otimes dx_{(\gamma\setminus i)\cup j \cup l},$$

which is zero because the summand with $j < l$ cancels with the summand with $l < j$. Finally, $T_2 = 0$ since $i \in \gamma$. $\square$

Set $\overline{\Omega}^\chi_k = \Omega^\chi_k / \ker \delta^\chi_k$.

3.10 Proposition. *Assume* $\operatorname{ht}\chi = 0$ *and let* $0 \leq k \leq n$.

(1) *The sequence* $0 \to \Omega^\chi_0 \xrightarrow{\delta^\chi_0} \Omega^\chi_1 \xrightarrow{\delta^\chi_1} \cdots \xrightarrow{\delta^\chi_{n-1}} \Omega^\chi_n \to 0$ *is exact.*
(2) $\overline{\Omega}^\chi_k$ *has basis* $\{\overline{y^{(a)} \otimes dx_\gamma} \,|\, a \in A, \gamma \in \Gamma_k, j \notin \gamma\}$ *for any* $1 \leq j \leq n$ *satisfying* $\chi(D_j) \neq 0$.
(3) $\dim_F \overline{\Omega}^\chi_k = p^n \binom{n-1}{k}$.

Proof. First, for any $a \in A$, $\gamma \in \Gamma_k$, we have

$$\delta^\chi_{k+1}\delta^\chi_k(y^{(a)} \otimes dx_\gamma) = \sum_{\substack{i,j \notin \gamma \\ i \neq j}} (-1)^{|\gamma\cup i_{>j}|}(-1)^{|\gamma_{>i}|} y^{(a-\epsilon_i-\epsilon_j)} dx_{\gamma\cup\{i,j\}},$$

which is zero since $(-1)^{|\gamma\cup i_{>j}|}(-1)^{|\gamma_{>i}|}$ is $(-1)^{|_i\gamma_j|}$ for $i < j$ and $-(-1)^{|_i\gamma_j|}$ for $j < i$. Hence $\operatorname{im}\delta^\chi_k \subseteq \ker \delta^\chi_{k+1}$ and the sequence in (1) is a complex.

Fix $1 \leq j \leq n$ with $\chi(D_j) \neq 0$. Now $\dim_F \Omega^\chi_k = p^n\binom{n}{k}$ and the set in (2) has cardinality $p^n\binom{n-1}{k}$, so it suffices to prove this set is linearly independent (for then the remaining claims follow by induction on k).

Suppose $\sum_{\substack{a,\gamma \\ j\notin\gamma}} c_{a,\gamma}\overline{y^{(a)} \otimes dx_\gamma} = 0$ with $c_{a,\gamma} \in F$. Then

$$\sum_{\substack{a,\gamma \\ j\notin\gamma}} \sum_{\substack{i\notin\gamma \\ i\neq j}} c_{a,\gamma}(-1)^{|\gamma_{>i}|} y^{(a-\epsilon_i)} \otimes dx_{\gamma\cup i} + \sum_{\substack{a,\gamma \\ j\notin\gamma}} c_{a,\gamma}(-1)^{|\gamma_{>j}|} y^{(a-\epsilon_j)} \otimes dx_{\gamma\cup j} = 0.$$

From our definition of $y^{(b)}$ for $b \notin A$, we see that the second sum is a linear combination of (distinct) standard basis vectors, none of which appears in the first sum. Therefore, $c_{a,\gamma} = 0$ for all (a, γ). $\square$

For completeness, we state the corresponding result for the restricted case $\operatorname{ht}\chi = -1$ (i.e., $\chi = 0$) suppressing the character in the notation (so Ω^0_k becomes Ω_k, δ^0_k becomes δ_k, etc.).

3.11 Proposition.

(1) *The sequence* $0 \to \Omega_0 \xrightarrow{\delta_0} \Omega_1 \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{n-1}} \Omega_n \to 0$ *is a complex and* $\ker \delta_k / \operatorname{im} \delta_{k-1}$ *is a direct sum of* $\binom{n}{k}$ *copies of the trivial module* F $(0 \leq k \leq n)$.

(2) $\overline{\Omega}_k$ *has basis* $\cup_j \overline{B}_j$, *where* $B_j = \{y^{(a)} \otimes dx_\gamma \,|\, a \in A, \gamma \in \Gamma_k, j \notin \gamma, a_j \neq 0,$ *and for* $i < j$, $a_i = 0$ *if* $i \notin \gamma$ *and* $a_i = p-1$ *if* $i \in \gamma\}$.

(3) $\dim_F \overline{\Omega}_k = (p^n - 1)\binom{n-1}{k}$.

Proof. See [H, 2.2, 2.3] (cf. also [Sh, 2.1]). $\square$

The complex of 3.11(1) is called the *de Rham complex* for W. As the proposition states, its homology modules are direct sums of the trivial module F. We see from 3.10(1) that when $\operatorname{ht}\chi = 0$, the homology modules in the χ-version of the de Rham complex vanish. This vanishing might have been expected because the trivial module has character 0 and so cannot appear as a subquotient of a module having nonzero character.

3.12 Theorem. *Assume* $\operatorname{ht}\chi = 0$. *If* $0 < k < n$, *then* Ω_k^χ *has as unique proper* W*-submodule* $\ker \delta_k^\chi$, *while* Ω_0^χ *and* Ω_n^χ *have no proper* W*-submodules. In particular, the* W*-module* $\overline{\Omega}_k^\chi$ *is simple for* $0 \leq k < n$.

Proof. Assume the theorem has been proved for a particular $\chi \in W^*$ and let $\Phi \in \operatorname{Aut} W$. We prove the theorem for the character $\psi := \chi^\Phi$. It suffices to prove the first statement (noting that $\overline{\Omega}_0^\psi$ is nonzero by 3.10(3)). By 3.8 and 2.5(2), we have W-isomorphisms

$$\Omega_k^\psi \cong Z^\psi(\omega_k) \cong [Z^\chi(\omega_k)]^\Phi \cong (\Omega_k^\chi)^\Phi.$$

If k is either 0 or n, this shows that Ω_k^ψ has no proper submodules. Assume $0 < k < n$. By 3.10, $\ker \delta_k^\psi$ is proper, and so the above isomorphism shows that it is the unique proper submodule of Ω_k^ψ.

From the previous paragraph and in light of 1.2(2), we may assume that $\chi(D_i) = \delta_{in}$ $(1 \leq i \leq n)$.

We first argue that $\overline{\Omega}_k^\chi$ is simple for $0 \leq k < n$, and for this it is enough by 2.1 to show that this module is generated by each of its maximal vectors (using 3.10 to see that $\overline{\Omega}_k^\chi \neq 0$). Fix $0 \leq k < n$ and let $v \in \overline{\Omega}_k^\chi$ be a maximal vector. According to 3.10(2) v can be expressed in the form $v = \sum_{\substack{a,\gamma \\ n\notin\gamma}} c_{a,\gamma} \overline{y^{(a)} \otimes dx_\gamma}$ with $c_{a,\gamma} \in F$. For $i < n$, we have (3.7)

$$0 = x_i D_n \cdot v = \sum_{\substack{a,\gamma \\ n\notin\gamma}} c_{a,\gamma}(a_i + 1)\overline{y^{(a-\epsilon_n+\epsilon_i)} \otimes dx_\gamma}.$$

Therefore, $c_{a,\gamma} = 0$ if $a_i \neq p-1$ for some $i < n$. Next, if $1 \leq i < j < n$, then

$$0 = x_i D_j \cdot v = \sum_{\substack{a,\gamma \\ i,n\notin\gamma \\ j\in\gamma}} c_{a,\gamma}(-1)^{|_j\gamma_i|}\overline{y^{(a)} \otimes dx_{(\gamma\setminus j)\cup i}}.$$

Therefore, $c_{a,\gamma} = 0$ if there exist $1 \le i < j < n$ with $j \in \gamma$, $i \notin \gamma$. Now

$$0 = x^{(2\epsilon_n)} D_n \cdot v = \sum_{\substack{a,\gamma \\ n \notin \gamma}} c_{a,\gamma} \binom{a_n + 1}{2} \overline{y^{(a+\epsilon_n)} \otimes dx_\gamma},$$

so $c_{a,\gamma} = 0$ if $a_n \notin \{0, p-1\}$.

Summarizing, we have

$$v = c_{\tau,\alpha} \overline{y^{(\tau)} \otimes dx_\alpha} + c_{b,\alpha} \overline{y^{(b)} \otimes dx_\alpha},$$

where $\tau = (p-1, \dots, p-1)$, $b = (p-1, \dots, p-1, 0)$, $\alpha = (1, 2, \dots, k)$.

Assume $k < n-1$. Then

$$0 = x^{(2\epsilon_n)} D_{n-1} \cdot v = c_{b,\alpha} \overline{y^{(b-\epsilon_{n-1}+2\epsilon_n)} \otimes dx_\alpha},$$

implying $c_{b,\alpha} = 0$. Therefore, $v = c_{\tau,\alpha} \overline{y^{(\tau)} \otimes dx_\alpha}$. Since $y^{(\tau)} \otimes dx_\alpha$ generates Ω_k^χ, it follows that v generates $\overline{\Omega}_k^\chi$.

Finally, we consider the case $k = n-1$. Since v is a weight vector and $\overline{y^{(\tau)} \otimes dx_\alpha}$ and $\overline{y^{(b)} \otimes dx_\alpha}$ are weight vectors having distinct weights, we have that either $c_{\tau,\alpha} = 0$ or $c_{b,\alpha} = 0$. If $c_{b,\alpha} = 0$, then v generates $\overline{\Omega}_k^\chi$ (using the argument above), so assume $c_{\tau,\alpha} = 0$. By 3.10(1), δ_{n-1}^χ induces an isomorphism $\overline{\Omega}_{n-1}^\chi \to \Omega_n^\chi$, which sends v to $c_{b,\alpha} y^{(\tau)} \otimes dx_{\alpha \cup n}$. Since this last vector generates Ω_n^χ, we conclude that v generates $\overline{\Omega}_{n-1}^\chi$. In each case, we have seen that v generates $\overline{\Omega}_k^\chi$. This finishes the proof that $\overline{\Omega}_k^\chi$ is simple for $0 \le k < n$.

We now prove the first statement of the theorem (still assuming $\chi(D_i) = \delta_{in}$). $\Omega_0^\chi \cong \overline{\Omega}_0^\chi$ and $\Omega_n^\chi \cong \overline{\Omega}_{n-1}^\chi$ (3.10(1)), so these modules have no proper submodules. Assume $0 < k < n$. From 3.10(1) we see that Ω_k^χ has composition factors $\overline{\Omega}_k^\chi$ and $\ker \delta_k^\chi \cong \overline{\Omega}_{k-1}^\chi$. Therefore, to show that $\ker \delta_k^\chi$ is the unique proper submodule of Ω_k^χ, it suffices to show that Ω_k^χ has no submodule isomorphic to $\overline{\Omega}_k^\chi$. Assume to the contrary that $\overline{\Omega}_k^\chi \cong S \le \Omega_k^\chi$. In particular, S has a maximal vector of weight ω_k (since $\overline{y^{(\tau)} \otimes dx_\alpha}$ is such in $\overline{\Omega}_k^\chi$). Now $\Omega_k^\chi \cong Z^\chi(\omega_k)$ (3.8), and 2.4(2) says each maximal vector of weight ω_k in $Z^\chi(\omega_k)$ is of the form $1 \otimes m$ with $0 \ne m \in L_0(\omega_k)$. Since any such vector generates $Z^\chi(\omega_k)$, we get $\overline{\Omega}_k^\chi \cong S = \Omega_k^\chi$ contradicting, for instance, that Ω_k^χ has two composition factors. This finishes the proof. □

4. Conclusion

In this final section, we assemble the preceding results and state the main theorems. Fix $\chi \in W^*$ and assume $\operatorname{ht} \chi \le 1$.

4.1 Theorem. *If S is a simple $u(W_0, \chi)$-module, then $Z^\chi(S)$ has a unique maximal W-submodule.*

Proof. Let S be a simple $u(W_0, \chi)$-module. If S is not W_0-isomorphic to any $L_0(\omega_k)$ $(0 \le k \le n)$, then $Z^\chi(S)$ is simple (2.6) and the theorem follows. Suppose $S \cong L_0(\omega_k)$ for some $0 \le k \le n$. Since $L_0(\omega_k)$ is a restricted W_0-module, we have $\chi(W_0) = 0$, implying $\operatorname{ht} \chi \le 0$. If $\operatorname{ht} \chi = -1$, then $\chi = 0$ (the restricted case) and the claim is well known (see, for instance, [HN, remarks before 3.2]). If $\operatorname{ht} \chi = 0$, then $Z^\chi(S) = Z^\chi(\omega_k) \cong \Omega_k^\chi$ (3.8), and the theorem follows from 3.12. □

For a simple $u(W_0, \chi)$-module S, let $L^\chi(S)$ denote the quotient of $Z^\chi(S)$ by its unique maximal W-submodule (4.1). If $\operatorname{ht} \chi \le 0$, we write $L^\chi(L_0(\lambda))$ more simply as $L^\chi(\lambda)$ for $\lambda \in \Lambda$. For completeness, we begin the description of the simple modules with the well-known restricted case $\operatorname{ht} \chi = -1$ (i.e., $\chi = 0$) (again suppressing the character in the notation as in 3.11). The theorem is due to G. Shen.

4.2 Theorem.

(1) *There are p^n distinct (up to isomorphism) simple $u(W)$-modules. They are represented by $\{L(\lambda) \mid \lambda \in \Lambda\}$.*
(2) *$L(\lambda) \cong Z(\lambda)$ if and only if $\lambda \notin \{\omega_0, \dots, \omega_n\}$. For $0 \le k \le n$, we have $L(\omega_k) \cong \overline{\Omega}_k$.*
(3) *If $\lambda \in \Lambda$ is not exceptional, then $\dim_F L(\lambda) = p^n \dim_F L_0(\lambda)$, while $\dim_F L(\omega_k) = \binom{n-1}{k}(p^n - 1)$ for $0 \le k < n$, and $L(\omega_n)$ is the one-dimensional trivial module.*

Proof. See [Sh, Theorem 2.1]. Note that 2.6 gives one direction in the first statement of (2). The proof in [Sh] of the second statement of (2) depends heavily on the grading of the module Ω_k. For a proof that does not use the grading and that is more along the lines of the proof of 3.12, see [N, Theorem 2.5.3]. □

4.3 Theorem. *Assume* $\operatorname{ht} \chi = 0$.

(1) *There are $p^n - 1$ distinct (up to isomorphism) simple $u(W, \chi)$-modules. They are represented by $\{L^\chi(\lambda) \mid \lambda \in \Lambda, \lambda \ne \omega_n\}$. We have $L^\chi(\omega_n) \cong L^\chi(\omega_{n-1})$.*
(2) *$L^\chi(\lambda) \cong Z^\chi(\lambda)$ if and only if $\lambda \notin \{\omega_1, \dots, \omega_{n-1}\}$. For $0 \le k < n$, we have $L^\chi(\omega_k) \cong \overline{\Omega}_k^\chi$.*
(3) *If $\lambda \in \Lambda$ is not exceptional, then $\dim_F L^\chi(\lambda) = p^n \dim_F L_0(\lambda)$, while*

$$\dim_F L^\chi(\omega_k) = \begin{cases} p^n \binom{n-1}{k}, & 0 \le k < n \\ p^n, & k = n. \end{cases}$$

Proof. (2) If $\lambda \in \Lambda$ is not exceptional, then $L^\chi(\lambda) \cong Z^\chi(\lambda)$ by 2.6. For $0 \le k \le n$, 3.8 says $Z^\chi(\omega_k) \cong \Omega_k^\chi$, and this latter module is simple if and only if $k \in \{0, n\}$ by 3.12. Moreover, for $0 \leq k < n$, 3.12 says $\overline{\Omega}_k^\chi$ is simple, and, since $\overline{\Omega}_k^\chi$ is a homomorphic image of Ω_k^χ (and hence of $Z^\chi(\omega_k)$), we have $\overline{\Omega}_k^\chi \cong L^\chi(\omega_k)$.

(1) Since $\operatorname{ht}\chi = 0$, we have $\chi(W_0) = 0$. Therefore, every simple $u(W_0, \chi)$-module is restricted and hence isomorphic to $L_0(\lambda)$ for some $\lambda \in \Lambda^\chi = \Lambda$. It now follows from 2.2 that every simple $u(W, \chi)$-module is isomorphic to $L^\chi(\lambda)$ for some $\lambda \in \Lambda$.

Using part (2), 3.8, and 3.10(1), we have $L^\chi(\omega_n) \cong Z^\chi(\omega_n) \cong \Omega_n^\chi \cong \overline{\Omega}_{n-1}^\chi \cong L^\chi(\omega_{n-1})$. Therefore, it suffices to show that there are at least $p^n - 1$ pairwise nonisomorphic simple $u(W, \chi)$-modules. Since $u(W, \chi^\Phi) \cong u(W, \chi)$ for any $\Phi \in \operatorname{Aut} W$, it follows from 1.2(2) that we may assume $\chi(D_i) = \delta_{in}$ $(1 \leq i \leq n)$.

Let $\lambda \in \Lambda$ and assume λ is not exceptional. Then $L^\chi(\lambda) \cong Z^\chi(\lambda)$ by part (2). Now 2.4(2) says every maximal vector of $Z^\chi(\lambda)$ is of the form $1 \otimes m$, where m is a maximal vector of $L_0(\lambda)$. Since $L_0(\lambda)$ has a unique maximal vector (up to scalar multiple) and the weight of this vector is λ, we conclude that the same must be true for $L^\chi(\lambda)$. On the other hand (and here is where we use the assumption that $\chi(D_i) = \delta_{in}$), the proof of 3.12 shows that if $k < n-1$, then every maximal vector in $\overline{\Omega}_k^\chi$ is a scalar multiple of $\overline{y^{(\tau)} \otimes dx_\alpha}$, which is easily seen to have weight ω_k, while $\overline{\Omega}_{n-1}^\chi$ has two linearly independent maximal vectors, namely $\overline{y^{(\tau)} \otimes dx_\alpha}$ and $\overline{y^{(b)} \otimes dx_\alpha}$. In light of the fact that $L^\chi(\omega_k) \cong \overline{\Omega}_k^\chi$ for $0 \leq k < n$ (by part (2)), we deduce that the W-modules $L^\chi(\lambda)$ with $\lambda \in \Lambda$, $\lambda \neq \omega_n$ are pairwise nonisomorphic.

(3) The first part follows from the fact that $L^\chi(\lambda) \cong Z^\chi(\lambda)$ for nonexceptional λ. The second part follows from (2), 3.10(3), and (1). $\square$

4.4 Theorem. *Assume* $\operatorname{ht}\chi = 1$. *Let* $\{S \mid S \in \mathcal{S}\}$ *be a set of representatives for the isomorphism classes of simple* $u(W_0, \chi)$*-modules.*

(1) *There are* $|\mathcal{S}|$ *distinct (up to isomorphism) simple* $u(W, \chi)$*-modules. They are represented by* $\{L^\chi(S) \mid S \in \mathcal{S}\}$.
(2) *For each* $S \in \mathcal{S}$, *we have* $L^\chi(S) \cong Z^\chi(S)$.
(3) *We have* $\dim_F L^\chi(S) = p^n \dim_F S$ $(S \in \mathcal{S})$.

Proof. (2) This is the last statement of 2.6.

(1) By 1.2(3), 2.5(1), and part (2) of this theorem, we may assume $\chi(W_{-1}) = 0$. Then $u(W, \chi)$ is a graded algebra with grading induced by the grading on W and, as such, satisfies the assumptions on the algebra A in [HN] (the argument in Example 3 of Section 2 there carries over to $u(W, \chi)$). Then 3.2 of that paper establishes the claim. (Incidentally, the fact that each simple $u(W, \chi)$-module is isomorphic to $L^\chi(S)$ for some $S \in \mathcal{S}$ follows also from 2.2 above.)

(3) This is clear from (2). $\square$

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